A Proof of the Riemann Hypothesis using Rouché’s Theorem and an Infinite Subdivision of the Critical Strip.

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ABSTRACT. Rouché’s Theorem is applied to the critical strip to show that the only zeros of the Riemann functions $\xi(s)$ and $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$. A function $\eta(s)$ is introduced, regular and with the same zeros as $\xi(s)$ in the critical strip. An infinite subdivision of the critical strip is made to prove that $\xi(s)$ has no zeros within the interiors of the regions where $\frac{1}{2} < \sigma < 1$. The regions are separated by boundary curves, which are also zero-free, due to the harmonic properties of $\Re[\xi(s)]$ and $\Im[\xi(s)]$. The Riemann Hypothesis is proved to be true.

1 Introduction

The work of this paper is based mainly on the work of H.M. Edwards[1], although other books and papers have been studied; the more important are listed in the References.

The Riemann Zeta function $\zeta(s)$ is a function of the complex variable $s = \sigma + it$, defined in the half plane $\sigma > 1$ by the absolutely convergent series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Riemann[2] showed that $\zeta(s)$ extends by analytical continuation to the whole complex plane, with only a simple pole at $s = 1$, with residue 1. He showed that $\zeta(s)$ satisfies the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{1}{2} s \pi \Gamma(1-s) \zeta(1-s)$$

Riemann introduces the function $\xi(s)$ as:

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

F.R.Allen 18th August 2017
which satisfies the functional equation
\[ \xi(s) = \xi(1 - s) \]
and has the same zeros as \( \zeta(s) \).

2 The function \( \xi(s) \)

Riemann obtains the equation
\[
\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \int_{1}^{\infty} \psi(x) \left( x^{\frac{s}{2}} + x^{\frac{(1-s)}{2}} \right) \frac{dx}{x}
\]
where \( \psi(x) \) is the theta function
\[
\psi(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}
\]

Let \( lx = \frac{1}{2} \log_e x \) and \( s = \sigma + it = \frac{1}{2} + y + it \). Use of a local coordinate \( y = \sigma - \frac{1}{2} \) in the critical strip, simplifies subsequent analysis.

It follows that
\[
\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \int_{1}^{\infty} 2\psi(x) (\cosh lx.y \cos lx.t + \sinh lx.y \sin lx.t)x^{-\frac{3}{4}} dx
\]
\[
= \frac{1}{2} + s(s-1)(I_1 + iI_2)
\]
where
\[
I_1 = \int_{1}^{\infty} \psi(x) (\cosh lx.y \cos lx.t)x^{-\frac{3}{4}} dx
\]
and
\[
I_2 = \int_{1}^{\infty} \psi(x) (\sinh lx.y \sin lx.t)x^{-\frac{3}{4}} dx
\]

With the definitions: \( U = y^2 - t^2 - \frac{1}{4} \) and \( V = 2yt \),
\[
s(s-1) = (\frac{1}{2} + y + it)(-\frac{1}{2} + y + it) = y^2 - t^2 - \frac{1}{4} + i.2yt
\]
\[
= U + iV
\]
With the change of variable \( u = \frac{1}{2} \log x \)

\[
I_1 = 2 \int_0^\infty \psi(e^{2u}) e^{\frac{u}{2}} \cosh(yu) \cos(tu) du
\]

and

\[
I_2 = 2 \int_0^\infty \psi(e^{2u}) e^{\frac{u}{2}} \sinh(yu) \sin(tu) du
\]

Since

\[
\xi(s) = \frac{1}{2} + (U + iV)(I_1 + iI_2)
\]

then

\[
\xi(s) = \frac{1}{2} + UI_1 - VI_2 + i(VI_1 + UI_2)
\]

These equations are fundamental in the following analysis.

3 The zeros of \( \xi(s) \) and the Riemann Hypothesis

By definition, the zeros of \( \xi(s) \) coincide with those of \( \zeta(s) \) in the critical strip defined as \( 0 \leq \sigma \leq 1 \) and all \( t \). The Riemann Hypothesis is that all the roots of \( \xi(s) \) lie on the line \( \sigma = \frac{1}{2} \). The trivial zeros of \( \zeta(s) \) occur when \( s \) is a negative even integer.

Hardy first showed that \( \zeta(s) \) has an infinity of zeros on the critical line \( RL_s = \frac{1}{2} \). Hadamard and de la Vallée Poussin proved independently that \( \zeta(s) \) has no zeros on the line \( RL_s = 1 \) or \( y = \frac{1}{2} \) in the notation of this paper. Hence, \( \xi(s) \) also has no zeros on \( RL_s = 1 \). Their work is described at length by Titchmarsh in Reference [4].

From the above equation for \( \xi(s) \)

\[
RL[\xi(s)] = \frac{1}{2} + UI_1 - VI_2
\]

\[
Im[\xi(s)] = VI_1 + UI_2
\]

It follows that necessary and sufficient conditions for \( \xi(s) \) to be zero are:

\[
I_1 = I_{1*} \quad \text{and} \quad I_2 = I_{2*}
\]

where

\[
I_{1*} = \frac{-U}{2(U^2 + V^2)} \quad \text{and} \quad I_{2*} = \frac{V}{2(U^2 + V^2)}
\]
As $t$ increases, $I_{1*} \to \frac{1}{2t^2}$ and $I_{2*} \to \frac{y}{t^3}$

For a given $y$ in the critical half-strip, $0 < y < \frac{1}{2}$, $Rl[\xi(s)]$, and $Im[\xi(s)]$ have individual zeros which are interleaved. No calculations have so far shown that $Rl[\xi(s)]$ and $Im[\xi(s)]$ can be zero simultaneously in the critical strip.

From their definitions, $I_1$ and $I_{1*}$ are even functions of $y$ and $t$, whilst $I_2$ and $I_{2*}$ are odd functions of $y$ and $t$. It follows that, if a zero of $\xi(s)$ exists within the critical strip, there must also exist three more zeros which are reflections in the axes $y = 0$ and $t = 0$.

In this paper, the critical strip is defined as the quarter strip $0 < y < \frac{1}{2}$, $t > 0$.

To prove the Riemann Hypothesis it is necessary and sufficient to show that there are no zeros of $\xi(s)$ in this particular choice of reduced critical strip.

4 Rouché’s Theorem and Application.

4.1 Rouché’s Theorem

This theorem[3] states that, if $f(s)$ and $g(s)$ are two functions, regular within and on a closed contour C on which $f(s)$ does not vanish and also $|g(s)| < |f(s)|$ on C, then $f(s)$ and $f(s) + g(s)$ have the same number of zeros within C.

The theorem can be used to prove that two functions have the same number of zeros within a given contour C. This suggests a method of proving the Riemann hypothesis when $f(s) + g(s) = \xi(s)$ and $f(s)$ is chosen to have no zeros on or within C.

Define $g(s) = \xi(s) - f(s)$, then $f(s)$ and $f(s) + g(s) = \xi(s)$ will have the same number of zeros within C if the inequality $|g(s)| < |f(s)|$ is true on C. We refer to this inequality as the Rouché Inequality.

4.2 Application to the Critical Strip.

If $C$ is a contour enclosing a region of the critical strip $0 < y < \frac{1}{2}$, $0 \leq t \leq \infty$ and $f(s) = Rlf + iImf$, then the Rouché Inequality is $|\xi(s) - f(s)| < |f(s)|$.

This is equivalent to

$$ (Rl\xi - Rlf)^2 + (Im\xi - Imf)^2 < Rlf^2 + Imf^2 $$

F.R.Allen 18th August 2017
which is

\[ Rl\xi^2 + Im\xi^2 < 2(Rl\xi.Rlf + Im\xi.Imf) \]

It is necessary to find a function \( f(s) \), regular within and on \( C \), which does not vanish on \( C \) and satisfies the Rouché Inequality. A range of choices for \( f(s) \) has been examined but no suitable function has been found to treat the situation when the contour \( C \) encloses the complete critical strip.

There are a number of ways in which the critical strip can be subdivided so that Rouché’s Theorem can be applied to each subdivision separately. In the next section functions \( f_1(s) \) and \( f_2(s) \) are introduced, which satisfy the Rouché Inequality in alternate subdivisions of the complete critical strip.

5 Functions \( \eta(s), f_1(s), f_2(s) \) and \( F(s) \)

5.1 A new function with the same zeros as \( \xi(s) \)

In Section 3 the necessary and sufficient conditions for a zero of \( \xi(s) \) are shown to be:

\[ I_1 = I_{1*} \quad \text{and} \quad I_2 = I_{2*} \]

where \( I_{1*} \) and \( I_{2*} \) are simple functions of \( y \) and \( t \).

A new regular function \( \eta(s) \) is defined as: \( \eta(s) = I_{1*} - I_1 + i(I_{2*} - I_2) \) which is now used in place of \( \xi(s) \) for the application of Rouché’s Theorem. It has the same zeros as \( \xi(s) \) in the critical strip.

5.2 The Rouché inequality with \( \eta(s) \) in place of \( \xi(s) \).

Define \( g(s) = \eta(s) - f(s) \), where \( f(s) \) is a regular function with no zeros on or within a contour \( C \), chosen to satisfy the condition \( |g(s)| < |f(s)| \) on \( C \).

This definition of \( g(s) \) implies that \( f(s) + g(s) = f(s) + \eta(s) - f(s) = \eta(s) \)

It follows that \( \eta(s) \) and \( f(s) \) have the same number of zeros within \( C \), provided that the inequality \( |g(s)| < |f(s)| \) is satisfied, which is \( |\eta(s) - f(s)| < |f(s)| \) on contour \( C \).

\[ F.R.Allen \ 18th \ August \ 2017 \]
5.3 Functions \( f_1(s), f_2(s) \) and \( F(s) \).

If \( f(s) \) is chosen to equal \( f_1(s) = I_1 + i.I_2 \), the inequality becomes \( |-I_1 - i.I_2| < |I_1 + i.I_2| \) or

\[
I_1^2 + I_2^2 < I_1^2 + I_2^2
\]

Then, if a function \( F(s) \) is defined as

\[
F(s) = I_1^2 + I_2^2 - (I_2^2 + I_2^2)
\]

where \( s = \frac{1}{2} + y + it \), the inequality can be written as \( F(s) > 0 \).

In a similar way, if \( f(s) = f_2(s) = -I_1 - i.I_2 \), the inequality becomes \( F(s) < 0 \).

\( f_1(s) \) and \( f_2(s) \) are regular functions which do not vanish on the contour \( C \) to be defined in Section 6.2. They also have no zeros within the contour \( C \).

The reason for these definitions of \( f_1(s), f_2(s) \) and \( F(s) \) will be evident in the discussion of the sub-division of the critical strip and of the choice of contour \( C \) which follows.

6 Subdivision of the critical strip.

6.1 The Regions \( R_{i,i+1} \)

The function \( F(s) = I_1^2 + I_2^2 - (I_1^2 + I_2^2) \) defines a family of curves \( F_i(s) = 0 \) which subdivide the critical strip. \( F_i(s) = 0 \) is the member of the family chosen to pass through the \( i^{th} \) root of \( \xi(s) \) at \( t = \rho_i \) on the critical line \( y = 0 \). \( F_i(s) = 0 \) at the root \( \rho_i \) because \( I_1 = I_1 \) and \( I_2 = I_2 \) at any zero of \( \xi(s) \), as derived in Section 3. The chosen subdivision of the critical strip is illustrated in Figure1.

When \( t=0 \), \( I_1 > I_1 \) and \( I_2 > I_2 \). Both \( I_1 \) and \( I_1 \) are monotonic decreasing functions of \( t \). The sign of \( I_1 - I_1 \) alternates as \( t \) passes through the roots \( \rho_i \) on the line. Since the values of \( I_1 \) and \( I_1 \) are dominant, \( F(s) \) oscillates in a similar way in step with \( I_1 - I_1 \).

The first region \( R_{0,1} \) is defined by the \( y \) axis, the curve \( F_1(s) = 0 \) and segments of the lines \( y=0 \) and \( y = 1/2 \), which are intersected by the first two boundaries.

All other regions \( R_{i,i+1}, i = 1,2, ... \) are defined by the curves \( F_i(s) = 0 \) and \( F_{i+1}(s) = 0 \).
and the corresponding segments of the lines $y = 0$ and $y = \frac{1}{2}$.

$F(s)$ alternates in sign as $t$ increases from 0 to infinity. $F_i(s) > 0$ in region $R_{i,i+1}$ when $i$ is even and is $< 0$ when $i$ is odd.

6.2 Analysis of a typical region $R_{i,i+1}$.

For purposes of illustration, the region $R_{3,4}$ is taken as shown in Figure 1, in which the function $F(s)$ is $< 0$.

A contour $C$ is defined by the segments of the lines $y = 0$ and $y = \frac{1}{2}$, together with two curves taken infinitesimally close to $F_3(s) = 0$ and $F_4(s) = 0$, respectively. It follows that, since $F(s) < 0$ throughout the region, it will be satisfied on the chosen contour $C$. The definition of $C$ implies that the zeros of $\xi(s)$ at $t = \rho_3$ and $t = \rho_4$ are excluded from $C$.

As described in Sections 5.2 - 5.3, with the choice $f(s) = f_2(s)$, the Rouché Inequality is satisfied at all points on and within $C$. Hence, the functions $\eta(s)$ and $\xi(s)$ have no zeros within $C$. In a similar way, for regions in which $F(s) > 0$, with $f(s) = f_1(s)$, the Rouché Inequality is again satisfied.

6.3 The boundary curves $F_i(s) = 0$ and $F_{i,i+1}(s) = 0$.

The analysis so far, using a subdivison of the critical strip, proves the Riemann Hypothesis for the individual regions, but excluding the boundary curves on which $F_i(s) = 0$. It remains to show that no other zeros of $\xi(s)$ can exist on these curves which pass through the known zeros $t = \rho_i$ on the critical line.

The function $\xi(s)$ is analytic, hence $Rl[\xi(s)]$ and $Im[\xi(s)]$ are harmonic functions, both satisfying Laplace’s equation i.e. $Rl[\xi(s)]_{yy} + Rl[\xi(s)]_{tt} = 0$ and $Im[\xi(s)]_{yy} + Im[\xi(s)]_{tt} = 0$. It follows that $Rl[\xi(s)]$ and $Im[\xi(s)]$ cannot have local maxima or minima.

It is known that there are no zeros of $\xi(s)$ on the line $y = \frac{1}{2}$, segments of which form part of the contour $C$ in each region $R_{i,i+1}$. However, $Rl[\xi(s)]$ and $Im[\xi(s)]$ can each have a zero on $y = \frac{1}{2}$, but not simultaneously.

6.3.1 The region $R_{0,1}$

The region $R_{0,1}$ is special with one boundary curve as the $y$ axis, where $t=0$. It can be shown that there are no zeros of $\xi(s)$ on the $y$ axis.

F.R.Allen 18th August 2017
Figure 1: Subdivision of the critical strip and choice of Contour C in Region $R_{3,4}$ shown thus: - - - - . The boundary curves are formed by the family $F_i(s) = 0$, where $i = 1, 2, ...$
6.3.2 The regions $R_{i,i+1}$ for $i > 0$

There are three cases to consider for points on the line $y = 1/2$

(i) $R_l[\xi(s)] = 0$ and $Im[\xi(s)]$ non-zero
(ii) $R_l[\xi(s)]$ non-zero and $Im[\xi(s)] = 0$
(iii) $R_l[\xi(s)]$ and $Im[\xi(s)]$ are both non-zero.

In Case(i), if a zero of $\xi(s)$ were to exist at $y = y_1$ in $0 < y < 1/2$ on $F_i(s)$, then $Im[\xi(s)]$, which is non-zero at $y = 1/2$ on $F_i(s)$, would pass through a local maximum or minimum as $y$ decreases to 0, before becoming zero on the line $y=0$.

Case(ii) is analogous to Case(i) except that $R_l[\xi(s)]$ replaces $Im[\xi(s)]$. The same argument applies because $R_l[\xi(s)] = 0$ at the point where $F_i(s)$ intersects the critical line $y = 0$ at the zero $t = \rho_i$ of $\xi(s)$.

In Case(iii), the same argument can be used for either $R_l[\xi(s)]$ or $Im[\xi(s)]$ to show that no zero of $\xi(s)$ can be present on any boundary curve $F_i(s) = 0$ for $i=1,2, ...$ and $0 < y < 1/2$.

6.4 Conclusions from the analysis of Section 6.

It has been shown that there are no zeros of $\xi(s)$ in all the subdivisions of the critical strip, interior to the boundary curves $F_i(s) = 0$. Furthermore, it has been shown that there can be no zeros of $\xi(s)$ on the boundary curves other than on the critical line $y = 0$, from which it follows that the Riemann Hypothesis is true.

7 The limiting forms of $I_1$ and $I_2$ as $t \to \infty$.

7.1 Notation

The integrals defined in Section 2 are:

$$ I_1 = 2 \int_0^\infty \psi(e^{2u})e^{yu} \cosh(yu) \cos(tu)du \quad \text{and} \quad I_2 = 2 \int_0^\infty \psi(e^{2u})e^{yu} \sinh(yu) \sin(tu)du $$

for $0 < y < 1/2$ and $0 < t < \infty$, where

$$ \psi(x) = \sum_{n=1}^\infty e^{-n^2 \pi x} $$
Define
\[ f_1^{(0)}(u) = 2\psi(e^{2u})e^{\frac{y}{2}}\cosh(yu) \quad \text{and} \quad f_2^{(0)}(u) = 2\psi(e^{2u})e^{\frac{y}{2}}\sinh(yu) \]
then
\[ I_1 = \int_0^\infty f_1^{(0)}(u)\cos(tu)du \quad \text{and} \quad I_2 = \int_0^\infty f_2^{(0)}(u)\sin(tu)du \]

The differential coefficients with respect to \( u \) are:
\[ f_1^{(r)}(u) = \frac{d^r f_1^{(0)}(u)}{du^r} \quad \text{and} \quad f_2^{(r)}(u) = \frac{d^r f_2^{(0)}(u)}{du^r} \]
where \( r = 1, 2, \ldots \)

It is convenient to define
\[ I_{1,N} = 2\int_0^\infty f_1^{(N)}(u)\cos(tu)du \quad \text{and} \quad I_{2,N} = 2\int_0^\infty f_2^{(N)}(u)\sin(tu)du \]
for \( N=0,1,2,\ldots \) The upper or lower function applies, according as \( N \) is even or odd, respectively.

This notation implies that
\[ I_{10} = I_1 \quad \text{and} \quad I_{20} = I_2 \]

7.2 Series expansions of \( I_1 \) and \( I_2 \).

Integration by parts for \( t > 0 \) gives:
\[ I_{10} = \int_0^\infty f_1^{(0)}(u)\cos(tu)du \]
\[ = \left[f_1^{(0)}(u)\frac{\sin(tu)}{t}\right]_0^\infty - \frac{1}{t} \int_0^\infty f_1^{(1)}(u)\sin(tu)du \]
Now
\[ \lim_{u \to 0} \left[f_1^{(0)}(u)\frac{\sin(tu)}{t}\right] = 0 \]
and
\[ \lim_{u \to \infty} \left[f_1^{(0)}(u)\frac{\sin(tu)}{t}\right] = 0 \]

F.R.Allen 18th August 2017
Hence,
\[ I_{10} = -\frac{1}{t} \int_{0}^{\infty} f_{1}^{(1)}(u) \sin(tu) \, du \]

Similarly,
\[ I_{20} = \left[ f_{2}^{(0)}(u) \left( \frac{\cos(tu)}{-t} \right) \right]_{0}^{\infty} + \frac{1}{t} \int_{0}^{\infty} f_{2}^{(1)}(u) \cos(tu) \, du \]

In this case
\[ \lim_{u \to 0} \left[ f_{2}^{(0)}(u) \left( \frac{\cos(tu)}{-t} \right) \right] = \frac{1}{t} f_{2}^{(0)}(0) \]

and
\[ I_{20} = \frac{1}{t} f_{2}^{(0)}(0) + \frac{1}{t} \int_{0}^{\infty} f_{2}^{(1)}(u) \cos(tu) \, du \]

After repeated integration by parts to give three terms plus a remainder,
\[ I_{1} = -\frac{1}{t^2} f_{1}^{(1)}(0) + \frac{1}{t^4} f_{1}^{(3)}(0) - \frac{1}{t^6} f_{1}^{(5)}(0) - \frac{I_{16}}{t^6} \]

and
\[ I_{2} = -\frac{1}{t^3} f_{2}^{(2)}(0) + \frac{1}{t^5} f_{2}^{(4)}(0) - \frac{1}{t^7} f_{2}^{(6)}(0) + \frac{I_{28}}{t^8} \]

In general
\[ I_{1} = \sum_{n=1}^{N} \left( - \right)^{n} \frac{f_{1}^{(2n-1)}(0)}{t^{2n}} \left( - \right)^{N} \frac{I_{1,2N}}{t^{2N}} \quad \text{and} \quad I_{2} = \sum_{n=1}^{N} \left( - \right)^{n} \frac{f_{2}^{(2n)}(0)}{t^{2n+1}} \left( - \right)^{N+1} \frac{I_{2,2N+2}}{t^{2N+2}} \]

7.3 The functional dependence of \( f_{1}^{(r)}(0) \) and \( f_{2}^{(r)}(0) \) on the coordinate \( y \).

By repeated application of Leibnitz’s theorem for the successive differential coefficients of a product, in terms of the differential coefficients of its factors,
\[ f_{1}^{(r)}(0) = \sum_{k=0}^{r-1} \lambda_{k} y^{k} \]

with \( r \) odd and \( k \) even, and

\[ F.R.Allen 18th August 2017 \]

11
\[ f_2^{(r)}(0) = \sum_{k=1}^{r-1} \lambda_k^r y^k \]

with \( r \) even and \( k \) odd.

The coefficients \( \lambda_k^r \) are given by

\[ \lambda_k^r = -\binom{r}{k} \frac{1}{2^{r-k}}. \]

The final expressions for \( I_1 \) and \( I_2 \) are:

\[
I_1 = \sum_{n=1}^{N} (-)^n \sum_{k=0}^{2n-2} \lambda_k^{2n-1} y^k \frac{1}{t^{2n}} + (-)^N I_{1,2N} \frac{1}{t^{2N}}
\]

with \( k \) even and

\[
I_2 = \sum_{n=1}^{N} (-)^n \sum_{k=1}^{2n-1} \lambda_k^{2n} y^k \frac{1}{t^{2n+1}} + (-)^{N+1} I_{2,2N+2} \frac{1}{t^{2N+2}}
\]

with \( k \) odd.

When \( N=4 \) the expansions are:

\[
I_1 = \frac{1}{2} \frac{1}{t^2} - \left( \frac{1}{8} + \frac{3}{2} y^2 \right) \frac{1}{t^4} + \left( \frac{1}{32} + \frac{5}{4} y^2 + \frac{5}{2} y^4 \right) \frac{1}{t^6} - \left( \frac{1}{128} + \frac{21}{32} y^2 + \frac{35}{8} y^4 + \frac{7}{2} y^6 \right) \frac{1}{t^8} + I_{1,8} \frac{1}{t^8}
\]

and

\[
I_2 = \frac{y}{t^3} - \left( \frac{y}{2} + 2 y^3 \right) \frac{1}{t^5} + \left( \frac{3}{16} y + \frac{5}{2} y^3 + 3 y^5 \right) \frac{1}{t^7} - \left( \frac{1}{16} y + \frac{7}{4} y^3 + 7 y^5 + 4 y^7 \right) \frac{1}{t^9} - I_{2,10} \frac{1}{t^{10}}
\]

**7.4 Calculations of \( I_1 \) and \( I_2 \)**

Calculations show that \( I_1 \) and \( I_2 \) converge rapidly towards \( I_{1,*} \) and \( I_{2,*} \), as \( t \) increases for \( 0 \leq y \leq \frac{1}{2} \). This indicates that the necessary and sufficient conditions for a zero of \( \xi(s) \) are approached as \( t \to \infty \). The relationships between \( I_1, I_2 \) and \( I_{1,*}, I_{2,*} \) are described in the next section.

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*F.R.Allen 18th August 2017*
8 Series expansions of $I_{1*}$ and $I_{2*}$.

8.1 Power series in $\frac{1}{t}$ and Limits as $t \to \infty$

The functions $I_{1*}(s)$ and $I_{2*}(s)$, defined in Section 3, can be expressed as power series in $1/t$ with coefficients which are polynomials in the variable $y$. After some algebraic manipulation, it is found that,

$$I_{1*}(y,t) = \sum_{n=1}^{\infty} (-1)^n \frac{2n-1}{t^{2n}} \lambda_k y \frac{1}{t^{2n}}$$

with $k$ even

and

$$I_{2*}(y,t) = \sum_{n=1}^{\infty} (-1)^n \frac{2n-1}{t^{2n+1}} \lambda_k y \frac{1}{t^{2n+1}}$$

with $k$ odd.

These expansions are identical to those obtained for $I_1$ and $I_2$, except that after $N$ terms the remainders are different. This significant result implies that, if $I_1$ and $I_2$ are written in the form

$$I_1 = (N\text{terms})_1 + R_1(N) \quad \text{and} \quad I_2 = (N\text{terms})_2 + R_2(N)$$

and

$$I_{1*} = (N\text{terms})_{1*} + R_{1*}(N) \quad \text{and} \quad I_{2*} = (N\text{terms})_{2*} + R_{2*}(N)$$

then

$$(N\text{terms})_1 = (N\text{terms})_{1*} \quad \text{and} \quad (N\text{terms})_2 = (N\text{terms})_{2*}$$

It follows that

$$I_1 - I_{1*} = R_1(N) - R_{1*}(N) \quad \text{and} \quad I_2 - I_{2*} = R_2(N) - R_{2*}(N)$$

The remainders after $N$ terms, are,

$$R_1(N) = (-)^N I_{1,2N} \frac{1}{t^{2N}} \quad \text{and} \quad R_2(N) = (-)^{N+1} I_{2,2N+2} \frac{1}{t^{2N+2}}$$
\[ R_{1*}(N) = \sum_{n=N+1}^{\infty} (-)^n \sum_{k=0}^{2n-2} \lambda_k^{2n-1} y^k \frac{1}{t^{2n}} \quad \text{and} \quad R_{2*}(N) = \sum_{n=N+1}^{\infty} (-)^n \sum_{k=1}^{2n-1} \lambda_k^{2n} y^k \frac{1}{t^{2n+1}} \]

where \( k \) is even for \( R_{1*}(N) \) and is odd for \( R_{2*}(N) \).

As \( N \to \infty \) \( R_{1*}(N) \) and \( R_{2*}(N) \to 0 \).

From Section 7.1 the definitions of \( I_{1,N} \) and \( I_{2,N} \) are given in terms of infinite integrals. Hence, the remainders \( R_{1}(N) \) and \( R_{2}(N) \) can be written as:

\[
R_1(N) = (-)^N 2 \int_0^{\infty} f_1^{(2N)}(u)^{\cos t u} du \frac{1}{t^{2N}} \quad \text{and} \quad R_2(N) = (-)^N 2 \int_0^{\infty} f_2^{(2N+2)}(u)^{\sin t u} du \frac{1}{t^{2N+2}}
\]

for \( N=1,2,3 \ldots \) The upper or lower function applies, according as \( N \) is even or odd, respectively.

The Riemann-Lebesque theorem[7] implies that the two infinite integrals both tend to zero as \( t \) tends to infinity. It follows that \( (I_{1*} - I_1) \) and \( (I_{2*} - I_2) \) both tend to zero as \( t \) tends to infinity.

### 8.2 Limiting forms of \( (I_1 - I_{1*}) \) and \( (I_2 - I_{2*}) \).

For given values of \( y \) and \( t \),

\[
I_1(y,t) = I_{1*}(y,t) - R_{1*}(N) + R_1(N) \quad \text{and} \quad I_2(y,t) = I_{2*}(y,t) - R_{2*}(N) + R_2(N)
\]

In the limit, as \( N \to \infty \),

\[
I_1 = I_{1*} + \lim_{N \to \infty} (-)^N I_{1,2N} \frac{1}{t^{2N}} \quad \text{and} \quad I_2 = I_{2*} + \lim_{N \to \infty} (-)^N I_{2,2N+2} \frac{1}{t^{2N+2}}
\]

This can be written as

\[
I_1 = I_{1*} + R_1(\infty) \quad \text{and} \quad I_2 = I_{2*} + R_2(\infty).
\]

For \( s = \frac{1}{2} + y + it \) to be a zero of \( \xi(s) \) in the critical strip, the remainders \( R_1(\infty) \) and \( R_2(\infty) \) must be zero simultaneously.

\[ F.R.Allen 18th August 2017 \]
The functions $\eta(s)$ and $\xi(s)$.

The function $\eta(s)$ introduced in Section 5.1, can be expressed as:

$$\eta(s) = I_{1*} - I_1 + i(I_{2*} - I_2) = \lim_{N \to \infty} (-)^{N+1} \left( I_{1,2N} \frac{1}{l^2N} + i.I_{2,2N+2} \frac{1}{l^2N+2} \right)$$

Numerical integration of the infinite integrals shows rapid convergence as $N$ is increased.

Whereas the definition of $\eta(s)$ is

$$\eta(s) = I_{1*} - I_1 + i(I_{2*} - I_2),$$

the Riemann function $\xi(s)$ can be expressed in terms of $I_1, I_{1*}, I_2, I_{2*}$ as follows:

$$2.Rl[\xi(s)] = 1 - \frac{I_1I_{1*} + I_2I_{2*}}{(I_1)^2 + (I_{2*})^2} \quad \text{and} \quad 2.\text{Im}[\xi(s)] = \frac{I_1I_{2*} - I_{1*}I_2}{(I_1)^2 + (I_{2*})^2}$$

The necessary and sufficient conditions for both $\eta(s)$ and $\xi(s)$ to be zero are,

$$I_1 = I_{1*} \quad \text{and} \quad I_2 = I_{2*},$$

as shown earlier in Section 3 for $\xi(s)$.

It is possible to express either $\xi(s)$ or $\eta(s)$ as linear functions of the components of the other function, with coefficients $U, V$ or $I_{1*}, I_{2*}$, which are simple functions of $y$ and $t$, thus:

$$Rl[\xi(s)] = -U.Rl[\eta(s)] + V.\text{Im}[\eta(s)] \quad \text{and} \quad \text{Im}[\xi(s)] = -V.Rl[\eta(s)] - U.\text{Im}[\eta(s)],$$

and

$$\frac{1}{2} Rl[\eta(s)] = I_{1*}.Rl[\xi(s)] - I_{2*}.\text{Im}[\xi(s)] \quad \text{and} \quad \frac{1}{2} \text{Im}[\eta(s)] = I_{2*}.Rl[\xi(s)] + I_{1*}.\text{Im}[\xi(s)].$$
10 Conclusion

A regular function $\eta(s)$ has been introduced with the same zeros as $\xi(s)$ in the critical strip. The known zeros of $\xi(s)$ on the critical line $\sigma = \frac{1}{2}$ have been used to define a family of curves $F_i(s) = 0$ which subdivide the critical strip into an infinity of regions $R_{i,i+1}$, $i=0,1,2 \ldots$. Rouché’s Theorem is applied to each region in turn to prove that no zeros of $\xi(s)$ lie in these regions, excluding the boundary curves $F_i(s) = 0$. $R_{0,1}$ is a special region, with the $\sigma$ axis as one boundary on which there are no zeros of $\xi(s)$. Since $\xi(s)$ is a regular function of $s$, $\text{Re}[\xi(s)]$ and $\text{Im}[\xi(s)]$ are harmonic functions which both satisfy Laplace’s equation. It follows that $\xi(s)$ cannot be zero on a boundary curve except at $t=\rho_i$ on the critical line. The analysis shows that the zeros of $\xi(s)$ must all lie on the critical line $\sigma = \frac{1}{2}$, and that the Riemann Hypothesis is true.

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F.R.Allen 18th August 2017

16
References


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F.R.Allen 18th August 2017