A Proof of the Riemann Hypothesis using an Infinite Subdivision of the Critical Strip.

Frederick R. Allen

20th November 2017

ABSTRACT. Rouché’s Theorem is applied to the critical strip to show that the only zeros of the Riemann functions $\zeta(s)$ and $\xi(s)$ lie on the critical line $\sigma = \frac{1}{2}$. A function $\eta(s)$ is introduced, regular with the same zeros as $\xi(s)$ in the critical strip. An infinite subdivision of the critical strip is made to prove that $\eta(s)$, and hence $\xi(s)$, have no zeros within the interiors of the regions where $\frac{1}{2} < \sigma < 1$. The regions are separated by boundary curves, which are also shown to be zero-free. The Riemann Hypothesis is proved to be true.

1 Introduction.

The Riemann Zeta function $\zeta(s)$ is a function of the complex variable $s = \sigma + it$ defined in the half plane $\sigma > 1$ by the absolutely convergent series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$  \hspace{1cm} (1)

Riemann[2] showed that $\zeta(s)$ extends by analytical continuation to the whole complex plane, with only a simple pole at $s = 1$, with residue 1. He showed that $\zeta(s)$ satisfies the functional equation:

$$\zeta(s) = 2^s\pi^{s-1}\sin\frac{1}{2}s\pi\Gamma(1-s)\zeta(1-s)$$  \hspace{1cm} (2)

Riemann introduces the function $\xi(s)$ as:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$  \hspace{1cm} (3)

which satisfies the functional equation

$$\xi(s) = \xi(1-s)$$  \hspace{1cm} (4)

and has the same zeros as $\zeta(s)$. This paper is dependent on the work of H.M.Edwards[1] for the theory of the function $\xi(s)$.  

1
2 The function $\xi(s)$.

Riemann obtains the equation

$$\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \int_{1}^{\infty} \psi(x) \left( x^{\frac{s}{2}} + x^{\frac{(1-s)}{2}} \right) \frac{dx}{x}$$

where $\psi(x)$ is the theta function

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^{2} \pi x}.$$  

Let $lx = \frac{1}{2} \log_{e} x$ and $s = \sigma + it = \frac{1}{2} + y + it$. Use of a local coordinate $y = \sigma - \frac{1}{2}$ in the critical strip, simplifies subsequent analysis.

It follows that

$$\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \int_{1}^{\infty} 2\psi(x)(\cosh lx.y \cos lx.t + \sinh lx.y \sin lx.t)x^{-\frac{3}{4}}dx = \frac{1}{2} + s(s-1)(I_1 + iI_2)$$

where

$$I_1 = \int_{1}^{\infty} \psi(x)(\cosh lx.y \cos lx.t)x^{-\frac{3}{4}}dx$$

and

$$I_2 = \int_{1}^{\infty} \psi(x)(\sinh lx.y \sin lx.t)x^{-\frac{3}{4}}dx$$

With the definitions,

$$U = y^{2} - t^{2} - \frac{1}{4}, \quad V = 2yt,$$

$$s(s-1) = (\frac{1}{2} + y + it)(-\frac{1}{2} + y + it) = y^{2} - t^{2} - \frac{1}{4} + i.2yt = U + iV.$$  

If $u = \frac{1}{2} \log_{e} x$,

$$I_1 = 2 \int_{0}^{\infty} \psi(e^{2u})e^{\frac{u}{2}} \cosh (yu) \cos (tu)du$$

and

$$I_2 = 2 \int_{0}^{\infty} \psi(e^{2u})e^{\frac{u}{2}} \sinh (yu) \sin (tu)du$$

F.R.Allen 20th November 2017
Since
\[ \xi(s) = \frac{1}{2} + (U + iV)(I_1 + iI_2) \] (14)
then
\[ Rl[\xi(s)] = \frac{1}{2} + UI_1 - VI_2 \quad \text{and} \quad Im[\xi(s)] = VI_1 + UI_2. \] (15)

3 The zeros of \( \xi(s) \) and the Riemann Hypothesis.

By definition, the zeros of \( \xi(s) \) coincide with those of \( \zeta(s) \) in the critical strip defined as \( 0 \leq \sigma \leq 1 \) and all \( t \). The Riemann Hypothesis is that all the roots of \( \xi(s) \) lie on the line \( \sigma = \frac{1}{2} \). The trivial zeros of \( \zeta(s) \) occur when \( s \) is a negative even integer.

Hardy first showed that \( \zeta(s) \) has an infinity of zeros on the critical line \( Rls = \frac{1}{2} \). Hadamard and de la Vallee Poussin proved independently that \( \zeta(s) \) has no zeros on the line \( Rls = 1 \), or \( y = \frac{1}{2} \) in the notation of this paper. Hence, \( \xi(s) \) also has no zeros on \( Rls = 1 \). Their work is described at length by Titchmarsh in Reference [4].

If \( I_1^* \) and \( I_2^* \) are defined as,
\[ I_1^* = -\frac{U}{2(U^2 + V^2)} \quad \text{and} \quad I_2^* = \frac{V}{2(U^2 + V^2)}, \] (16)
the following theorem can be proved using Equations (15).

THEOREM 3.1 The necessary and sufficient conditions for \( \xi(s) \) to be zero in the Critical Strip are:
\[ I_1 = I_1^* \quad \text{and} \quad I_2 = I_2^* \] (17)

For a given \( y \) in the critical half-strip, \( 0 < y < \frac{1}{2} \), \( Rl[\xi(s)] \) and \( Im[\xi(s)] \) have individual zeros which are interleaved. No calculations have so far shown that \( Rl[\xi(s)] \) and \( Im[\xi(s)] \) can be zero simultaneously in the critical strip.

From their definitions, \( I_1 \) and \( I_1^* \) are even functions of \( y \) and \( t \), whilst \( I_2 \) and \( I_2^* \) are odd functions of \( y \) and \( t \). It follows that, if a zero of \( \xi(s) \) exists within the critical strip, there must also exist three more zeros which are reflections in the axes \( y = 0 \) and \( t = 0 \).

\[ F.R.Allen 20th November 2017 \]
In this paper, the critical strip is defined as the quarter strip $0 < y < \frac{1}{2}, \ t > 0$. To prove the Riemann Hypothesis it is necessary and sufficient to show that there are no zeros of $\xi(s)$ in this particular choice of reduced critical strip.

4 Rouché’s Theorem and Application.

4.1 Rouché’s Theorem.

This theorem[3] states that, if $f(s)$ and $g(s)$ are two functions, regular within and on a closed contour $C$ on which $f(s)$ does not vanish and also $|g(s)| < |f(s)|$ on $C$, then $f(s)$ and $f(s) + g(s)$ have the same number of zeros within $C$.

The theorem can be used to prove that two functions have the same number of zeros within a given contour $C$. This suggests a method of proving the Riemann hypothesis when $f(s) + g(s) = \xi(s)$ and $f(s)$ is chosen to have no zeros on or within $C$.

Define $g(s) = \xi(s) - f(s)$, then $f(s)$ and $f(s) + g(s) = \xi(s)$ will have the same number of zeros within $C$ if the inequality $|g(s)| < |f(s)|$ is true on $C$. We refer to this inequality as the Rouché Inequality.

4.2 Application to the Critical Strip.

If $C$ is a contour enclosing a region of the critical strip $0 < y < \frac{1}{2}, \ 0 \leq t \leq \infty$ and $f(s) = Rlf + i.Imf$, then the Rouché Inequality is $|\xi(s) - f(s)| < |f(s)|$.

This is equivalent to

$$(Rl\xi - Rlf)^2 + (Im\xi - Imf)^2 < Rlf^2 + Imf^2$$

which is

$$Rl\xi^2 + Im\xi^2 < 2(Rl\xi.Rlf + Im\xi.Imf)$$

It is necessary to find a function $f(s)$, regular within and on $C$, which does not vanish on $C$ and satisfies the Rouché Inequality. A range of choices for $f(s)$ has been examined but no suitable function has been found to treat the situation when the contour $C$ encloses the complete critical strip.

There are a number of ways in which the critical strip can be subdivided so that Rouché’s

\[ F.R.Allen 20th November 2017 \]
Theorem can be applied to each subdivision separately. In the next section functions \(f_1(s)\) and \(f_2(s)\) are introduced, which satisfy the Rouché Inequality in alternate subdivisions of the complete critical strip.

5 The functions \(\eta(s), f_1(s)\) and \(f_2(s)\).

An analytic function \(\eta(s)\) can be defined as:

\[
\eta(s) = I_{1s} - I_1 + i(I_{2s} - I_2)
\]  

(20)

This function is the sum of two analytic functions,

\[
f_1(s) = I_{1s} + iI_{2s} \quad \text{and} \quad f_2(s) = -(I_1 + iI_2).
\]  

(21)

\(\eta(s), f_1(s)\) and \(f_2(s)\) are regular functions in the Critical Strip.

From the definitions in Equations (12)-(13) and (16), the Cauchy-Riemann equations can be obtained as:

\[
\frac{\partial I_{1s}}{\partial y} = \frac{\partial I_{2s}}{\partial t}, \quad \frac{\partial I_{1s}}{\partial t} = -\frac{\partial I_{2s}}{\partial y} 
\]  

(22)

and

\[
\frac{\partial I_1}{\partial y} = \frac{\partial I_2}{\partial t}, \quad \frac{\partial I_1}{\partial t} = -\frac{\partial I_2}{\partial y}
\]  

(23)

The Cauchy-Riemann equations for \(\eta(s)\) then follow as,

\[
\frac{\partial \Re[\eta(s)]}{\partial y} = \frac{\partial \Im[\eta(s)]}{\partial t}, \quad \frac{\partial \Re[\eta(s)]}{\partial t} = -\frac{\partial \Im[\eta(s)]}{\partial y}
\]  

(24)

The necessary and sufficient conditions for \(\xi(s)\) to be zero are given in Equations(17) as

\[
I_1 = I_{1s} \quad \text{and} \quad I_2 = I_{2s}
\]  

(25)

It follows , from the definition of \(\eta(s)\) above, that \(\xi(s)\) and \(\eta(s)\) have the same zeros in the critical strip, including on the critical line \(y = 0\).
5.1 The Rouché inequality with $\eta(s)$ in place of $\xi(s)$.

Define $g(s) = \eta(s) - f(s)$, where $f(s)$ is a regular function with no zeros on or within a contour $C$, chosen to satisfy the condition $|g(s)| < |f(s)|$ on $C$.

This definition of $g(s)$ implies that $f(s) + g(s) = f(s) + \eta(s) - f(s) = \eta(s)$

It follows that $\eta(s)$ and $f(s)$ have the same number of zeros within $C$, provided that the inequality $|g(s)| < |f(s)|$ is satisfied, which is $|\eta(s) - f(s)| < |f(s)|$, on contour $C$.

5.2 The choice of the function $f(s)$.

If $f(s)$ is chosen to equal $f_1(s) = I_{1*} + i.I_{2*}$, the inequality becomes $|-I_1 - i.I_2| < |I_{1*} + i.I_{2*}|$

or

$I_1^2 + I_2^2 < I_{1*}^2 + I_{2*}^2$  \hspace{1cm} (26)

Then, if a function $F(s)$ is defined as

$$F(s) = I_{1*}^2 + I_{2*}^2 - (I_1^2 + I_2^2)$$  \hspace{1cm} (27)

where $s = \frac{1}{2} + y + it$, the inequality can be written as $F(s) > 0$.

In a similar way, if $f(s) = f_2(s) = -(I_1 + i.I_2)$, the inequality becomes $F(s) < 0$.

$f_1(s)$ and $f_2(s)$ are regular functions which do not vanish on the contour $C$ to be defined in Section 6.2. They also have no zeros within the contour $C$.

The reason for these definitions of $f_1(s)$, $f_2(s)$ and $F(s)$ will be evident in the discussion of the sub-division of the critical strip and of the choice of contour $C$ which follows.

6 Subdivision of the critical strip.

6.1 The Regions $R_{i,i+1}$.

The function $F(s) = I_{1*}^2 + I_{2*}^2 - (I_1^2 + I_2^2)$ defines a family of curves $F_i(s) = 0$ which subdivide the critical strip. $F_i(s) = 0$ is the member of the family chosen to pass through the $i^{th}$ root of $\xi(s)$ at $t = \rho_i$ on the critical line $y = 0$. $F_i(s) = 0$ at the root $\rho_i$ because $I_1 = I_{1*}$ and $I_2 = I_{2*}$ at any zero of $\xi(s)$, as shown in Section 3. The chosen subdivision of the critical strip is illustrated in Figure1.
When $t=0$, $I_{1*} > I_1$ and $I_{2*} > I_2$. $I_{1*}$ and $I_1$ are both monotonic decreasing functions of $t$ when $t > 7$. $I_2$ initially increases from zero to a maximum, before decreasing as $t$ increases. The sign of $I_{1*} - I_1$ alternates as $t$ passes through the roots $\rho_i$ on the line. Since the values of $I_{1*}$ and $I_1$ are dominant, the sign of $F(s)$ alternates in step with that of $I_{1*} - I_1$.

The first region $R_{0,1}$ is defined by the y axis, the curve $F_1(s) = 0$ and segments of the lines $y=0$ and $y = 1/2$, which are intersected by the first two boundaries.

All other regions $R_{i,i+1}$, $i = 1, 2, ...$ are defined by the curves $F_i(s) = 0$ and $F_{i+1}(s) = 0$ and the corresponding segments of the lines $y = 0$ and $y = 1/2$.

$F(s)$ alternates in sign as $t$ increases from 0 to infinity. $F(s)$ is $> 0$ in region $R_{i,i+1}$ when $i$ is even and is $< 0$ when $i$ is odd.

6.2 Analysis of a typical region $R_{i,i+1}$.

For purposes of illustration, the region $R_{3,4}$ is taken as shown in Figure 1, in which the function $F(s)$ is $< 0$. A contour $C_3$ is defined by the segments of the lines $y = 0$ and $y = 1/2$, together with two curves taken infinitesimally close to, but not coinciding with, the boundary curves $F_3(s) = 0$ and $F_4(s) = 0$, respectively. It follows that, since $F(s) < 0$ throughout the region, it will be satisfied on the chosen contour $C_3$. The definition of $C_3$ ensures that the zeros of $\xi(s)$ at $t = \rho_3$ and $t = \rho_4$ are excluded from $C_3$. 

_F.R.Allen 20th November 2017_
Contour $C_3$ is shown as: - - - - - -

Figure 1. Subdivision of the Critical Strip.
The arc of the contour \( C_3 \) adjacent to the boundary curve \( F_3(s) = 0 \), can be taken with a separation from \( F_3(s) = 0 \) of \( \epsilon > 0 \). If a zero of \( \xi(s) \) exists between the two curves, a new value of \( \epsilon, \epsilon_1 < \epsilon \), can be chosen to bring the zero within the contour \( C_3 \).

The arc of the contour \( C_3 \) adjacent to the boundary curve \( F_4(s) = 0 \), can be treated in the same way, to remove zeros which might exist between the boundary curve \( F_4(s) = 0 \) and the contour \( C_3 \).

As described in Section 5.2, with the choice \( f(s) = f_2(s) \), the Rouché Inequality is satisfied at all points on and within \( C_3 \). Hence, the functions \( \eta(s) \) and \( \xi(s) \) have no zeros within \( C_3 \). In a similar way, for regions in which \( F(s) > 0 \), with \( f(s) = f_1(s) \), the Rouché Inequality is again satisfied.

6.3 Conclusions from the application of Rouchés Theorem to the Critical Strip.

For each contour \( C_i \) in Region \( R_{i,i+1} \), Rouchés Theorem has been used to prove that no zeros of \( \xi(s) \) can exist within the Regions of the Critical Strip. The Riemann Hypothesis is proved for the Regions, but it remains to prove that there are no zeros on the boundary curves \( F_i(s) = 0, i = 1, 2, \ldots \).

7 The Boundary Curves \( F_i(s) = 0, i=1,2,\ldots \).

7.1 The function \( F_i(s) \).

In this paper the subdivision of the Critical Strip is made using the curves,

\[
F_i(s) = I_{1s}^2 + I_{2s}^2 - (I_1^2 + I_2^2) = 0 \tag{28}
\]

which intersect the Critical Line at the roots \( t = \rho_i \) of \( \xi(s) \) for \( i=1,2,\ldots \). It has been shown in Section 3 that the necessary and sufficient conditions for a zero of \( \xi(s) \) are

\[
I_1 = I_{1s} \quad \text{and} \quad I_2 = I_{2s} \tag{29}
\]

The condition \( F_i(s) = 0 \) on a boundary curve is a necessary, but not a sufficient condition for a zero to exist at a given point. The choice of boundary curves has been made so that the Rouché Inequality takes the form, \( F_i(s) > 0 \) or \( F_i(s) < 0 \), in alternate Regions of the Critical Strip. The contours \( C_i \) then lie within regions where \( F_i(s) > 0 \) or \( F_i(s) < 0 \). Rouchés Theorem has been used to show that there are no zeros of \( \eta(s) \), and therefore of \( \xi(s) \), within the contours \( C_i \).

---

F.R.Allen 20th November 2017

9
7.2 Proof that there are no zeros of \( \xi(s) \) on the Boundary Curves \( F_i(s) = 0 \).

7.2.1 Boundaries of the Region \( R_{0,1} \) defined by the contour \( C_0 \).

Region \( R_{0,1} \) is bounded by segments of the lines \( y=0 \) and \( y = \frac{1}{2} \) on which there are no zeros, together with the curves \( F_0(s) = 0 \) and \( F_1(s) = 0 \). The boundary curve \( F_1(s) \) is considered in the next Section as a member of the infinite family \( F_i(s) = 0 \) for \( i=1,2,\ldots \).

In the case of the special boundary \( F_0(s) = 0 \) on the \( y \) axis where \( t=0 \), from Equations(15),

\[
\xi(s) = \frac{1}{2} + (y^2 - \frac{1}{4})I_1(y,0)
\]

where

\[
0.01152 \leq I_1(y,0) \leq 0.01155 \quad \text{for} \quad 0 \leq y \leq 1/2.
\]

Hence, there are no zeros of \( \xi(s) \) on the \( y \) axis bounding Region \( R_{0,1} \), which is the boundary curve \( F_0(s) = 0 \).

7.2.2 The Boundaries of Regions \( R_{i,i+1} \) for \( i=1,2,\ldots \).

It remains to prove that there are no zeros on the boundary curves \( F_i(s) = 0 \) for \( i=1,2,\ldots \).

It is convenient to define

\[
\alpha = Re[\xi(s)] \quad \text{and} \quad \beta = Im[\xi(s)]
\]

From Equations(15) for \( \alpha \) and \( \beta \), it follows that,

\[
|\xi(s)|^2 = \alpha^2 + \beta^2 = \alpha + (U^2 + V^2)(I_{1*}^2 + I_{2*}^2) - \frac{1}{4}
\]

The definitions of \( I_{1*} \) and \( I_{2*} \) in Equations(16) give,

\[
I_{1*}^2 + I_{2*}^2 = 1/4(U^2 + V^2)
\]

Equation(33) becomes,

\[
\alpha^2 + \beta^2 = \alpha + (I_{1*}^2 + I_{2*}^2)/4(I_{1*}^2 + I_{2*}^2) - \frac{1}{4}
\]

On a boundary curve, \( F_i(s) = I_{1*}^2 + I_{2*}^2 - (I_1^2 + I_2^2) = 0 \),

therefore

\[
\alpha^2 + \beta^2 = \alpha
\]

F.R.Allen 20th November 2017
This equation shows that the real part of $\xi(s)$, $\alpha$, is always $\geq 0$ on the boundary curves $F_i(s) = 0$. But it is known[1] that there are no zeros of $\xi(s)$ when $y = \frac{1}{2}$; therefore, $\alpha$ and $\beta$ cannot both be zero when $y = \frac{1}{2}$.

From Equation (36), $\alpha > 0$ at the intersection of a boundary curve with the line $y = \frac{1}{2}$. If $\alpha$ were zero, $\beta$ would also be zero, which is not possible, since there are no zeros of $\xi(s)$ on the line $y = \frac{1}{2}$.

If a zero of $\xi(s)$ were present at $y = y_0$ on a boundary curve, the function $\alpha$, which is a continuous function of $y$ and is $> 0$ at $y=1/2$, becomes zero at $y = y_0$, then takes negative values before increasing to zero on the critical line $y = 0$.

This requires $\alpha$ to become negative in the interval $0 < y < y_0$, but Equation(36) shows this to be impossible on a boundary curve. Therefore, a zero of $\xi(s)$ cannot be present on a boundary curve.

7.2.3 Conclusion from the analysis of boundary curves.

The analysis proves that no zeros of $\xi(s)$ can occur on the boundary curves of all regions $R_{i,i+1}$, $i = 0, 1, 2, ...$ of the chosen subdivision of the Critical Strip.

8 Conclusions.

An analytic function $\eta(s)$ has been introduced with the same zeros as $\xi(s)$, which is regular in the critical strip. The known zeros of $\xi(s)$ on the critical line $\sigma = \frac{1}{2}$ have been used to define a family of curves $F_i(s) = 0$ which subdivide the critical strip into the regions $R_{i,i+1}$, $i=0,1,2, ...$ Rouché’s Theorem is applied to each region in turn to prove that no zeros of $\eta(s)$, and therefore of $\xi(s)$, lie in these regions, excluding the boundary curves $F_i(s) = 0$. $R_{0,1}$ is a special region, with the $\sigma$ axis as one boundary on which there are no zeros of $\xi(s)$. It is proved that $\xi(s)$ cannot be zero on a boundary curve except at $t=\rho_i$ on the critical line. The analysis in this paper shows that the zeros of $\xi(s)$ must all lie on the critical line $\sigma = \frac{1}{2}$, and that the Riemann Hypothesis is true.
References


F.R.Allen 20th November 2017