A compact Proof of the Riemann Hypothesis using the Riemann function $\xi(s)$ in terms of two infinite integrals and two related functions of the coordinates $(\sigma, t)$, within the Critical Strip.

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ABSTRACT. Two infinite integrals, associated with the Riemann $\xi(s)$ function, together with two related functions of the coordinates $(\sigma, t)$, are used to define a family of curves throughout the Critical Strip. It is shown that, by definition, no zeros of $\xi(s)$ can lie on these curves, except for a subset of the curves which intersect the Critical Line at the known zeros of $\xi(s)$. It is also proved that, the only zeros which can lie on this subset of curves, must fall on the Critical Line. The Riemann Hypothesis is proved to be true.

1 Introduction.

The Riemann Zeta function $\zeta(s)$ is a function of the complex variable $s = \sigma + it$ defined in the half plane $\sigma > 1$ by the absolutely convergent series:

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$$

(1)

Riemann[2] showed that $\zeta(s)$ extends by analytical continuation to the whole complex plane, with only a simple pole at $s = 1$, with residue 1. He showed that $\zeta(s)$ satisfies the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{1}{2}s\pi \Gamma(1 - s)\zeta(1 - s)$$

(2)

Riemann introduces the function $\xi(s)$ as:

$$\xi(s) = \frac{1}{2}s(s - 1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

(3)
which satisfies the functional equation

\[ \xi(s) = \xi(1 - s) \]  

and has the same zeros as \( \zeta(s) \). This paper is dependent on the work of H.M.Edwards[1] for the theory of the function \( \xi(s) \).

2 The function \( \xi(s) \).

Riemann[1] obtains the equation

\[ \xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \int_{1}^{\infty} \psi(x)(x^s + x^{1-s}) \frac{dx}{x} \]  

where \( \psi(x) \) is the theta function

\[ \psi(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}. \]  

Let \( lx = \frac{1}{2} \log_e x \) and \( s = \sigma + it = \frac{1}{2} + y + it \). Use of a local coordinate \( y = \sigma - \frac{1}{2} \) in the critical strip, simplifies subsequent analysis.

It follows that

\[ \xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \int_{1}^{\infty} 2\psi(x)(\cosh lx.y \cos lx.t + \sinh lx.y \sin lx.t)x^{-\frac{3}{4}} dx \]

\[ = \frac{1}{2} + s(s-1)(I_1 + iI_2) \]  

where

\[ I_1 = \int_{1}^{\infty} \psi(x)(\cosh lx.y \cos lx.t)x^{-\frac{3}{4}} dx \]  

and

\[ I_2 = \int_{1}^{\infty} \psi(x)(\sinh lx.y \sin lx.t)x^{-\frac{3}{4}} dx \]  

With the definitions,

\[ U = y^2 - t^2 - \frac{1}{4}, \quad V = 2yt, \]

\[ s(s-1) = (\frac{1}{2} + y + it)(-\frac{1}{2} + y + it) = y^2 - t^2 - \frac{1}{4} + i.2yt = U + iV. \]  

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If \( u = \frac{1}{2} \log x \),
\[
I_1 = 2 \int_0^\infty \psi(e^{2u}) e^{\frac{uy}{2}} \cosh yu \cos tu \, du
\] (12)
and
\[
I_2 = 2 \int_0^\infty \psi(e^{2u}) e^{\frac{uy}{2}} \sinh yu \sin tu \, du
\] (13)
Since
\[
\xi(s) = \frac{1}{2} + (U + iV)(I_1 + iI_2)
\] (14)
then
\[
\xi(s) = \frac{1}{2} + UI_1 - VI_2 + i(VI_1 + UI_2)
\] (15)
These equations are fundamental in the following analysis.

3 The zeros of \( \xi(s) \) and the Riemann Hypothesis.

By definition, the zeros of \( \xi(s) \) coincide with those of \( \zeta(s) \) in the critical strip defined as \( 0 \leq \sigma \leq 1 \) and all \( t \). The Riemann Hypothesis is that all the roots of \( \xi(s) \) lie on the line \( \sigma = \frac{1}{2} \). The trivial zeros of \( \zeta(s) \) occur when \( s \) is a negative even integer

Hardy first showed that \( \zeta(s) \) has an infinity of zeros on the Critical Line \( Rl_s = \frac{1}{2} \). Hadamard and de la Vallée Poussin proved independently that \( \zeta(s) \) has no zeros on the line \( Rl_s = 1 \), or \( y = \frac{1}{2} \) in the notation of this paper. Hence, \( \xi(s) \) also has no zeros on \( Rl_s = 1 \). Their work is described at length by Titchmarsh in Reference [4].

From the Equation(15) for \( \xi(s) \),
\[
Rl[\xi(s)] = \frac{1}{2} + UI_1 - VI_2 \quad \text{and} \quad Im[\xi(s)] = VI_1 + UI_2.
\] (16)
If \( I_1^* \) and \( I_2^* \) are defined as,
\[
I_1^* = \frac{-U}{2(U^2 + V^2)} \quad \text{and} \quad I_2^* = \frac{V}{2(U^2 + V^2)},
\] (17)
then, from Equations(16)-(17), the following theorem can be proved.

THEOREM 3.1 The necessary and sufficient conditions for \( \xi(s) \) to be zero are:
\[
I_1 = I_1^* \quad \text{and} \quad I_2 = I_2^*
\] (18)

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in the Critical Strip.

For a given $y$ in the Critical Half-strip, $0 < y < \frac{1}{2}$, $Rl[\xi(s)]$, and $Im[\xi(s)]$ have individual zeros which are interleaved. No calculations have so far shown that $Rl[\xi(s)]$ and $Im[\xi(s)]$ can be zero simultaneously in the Critical Strip.

From their definitions, $I_1$ and $I_{1*}$ are even functions of $y$ and $t$, whilst $I_2$ and $I_{2*}$ are odd functions of $y$ and $t$. It follows that, if a zero of $\xi(s)$ exists within the Critical Strip, there must also exist three more zeros which are reflections in the axes $y = 0$ and $t = 0$.

In this paper, the Critical Strip is defined as the quarter strip $0 < y < \frac{1}{2}$, $t > 0$.

To prove the Riemann Hypothesis it is necessary and sufficient to show that there are no zeros of $\xi(s)$ in this particular choice of reduced Critical Strip.

### 4 The values of $\xi(s)$ on the Critical Line.

As discussed in Section 3, $Rl[\xi(s)]$ has an infinite number of roots on the Critical Line. When $y=0$ and $t=0$, $Rl[\xi(s)] = 0.4971..$ and decreases as $t$ increases, to become zero at the first root $\rho_1 = 14.1347..$, where it changes sign from positive to negative.

Thereafter, it changes from negative to positive at $t=\rho_i$ when $i$ is even, or from positive to negative at $t=\rho_i$, when $i$ is odd. $Im[\xi(s)]$ is zero at all points on the Critical Line $y=0$.

At a zero of $Rl\xi(s)$ on $y=0$, Equations(16) give,

$$0 = \frac{1}{2} - (\rho_i^2 + \frac{1}{4})I_1(0, \rho_i)$$

(19)

From the definition of $I_{1*}$ in Equations(17),

$$I_{1*}(0, \rho_i) = 1/2(\rho_i^2 + \frac{1}{4})$$

(20)

then

$$I_1(0, \rho_i) = I_{1*}(0, \rho_i)$$

(21)

Since also

$$I_2(0, \rho_i) = I_{2*}(0, \rho_i),$$

it follows from THEOREM 3.1 that $\xi(s)$ has a zero at the point $(0, \rho_i)$.

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At points \((0,t)\) between roots,
\[
Rl[\xi(s)] = \frac{1}{2} - (t^2 + \frac{1}{4})I_1(0, t)
\]
where, from Equation(12),
\[
I_1(0, t) = 2 \int_0^\infty \psi(e^{2u})e^{\frac{u}{2}} \cos(tu)du
\]

\[ (22) \]
\[ (23) \]

5 Definition of a family of curves in the Critical Strip.

5.1 Boundary Curves.

A family of curves is defined as:
\[
f_i(y, t) = I_1(y, t) - I_1 = 0
\]
which intersect the Critical Line at the known roots \(\rho_i\) of \(\xi(s)\), for \(i=1,2,3,...\), to produce an infinite subdivision of the Critical Strip into regions \(R_{i,i+1}\) for \(i=0,1,2,...\).

The curves intersect the boundary \(y=\frac{1}{2}\) at \(t=\tau_i\), where \(\tau_i < \rho_i\), since the curves have a small negative curvature. For example, \(\rho_1 = 14.134725\) and \(\tau_1 = 13.949366\).

This family of individual curves will be referred to as Boundary Curves, which define the subdivision of the Critical Strip. Figure 1 shows the subdivision obtained using the curves \(f_1(y, t) = 0\) of Equation(24). The small negative curvature of the curves is not shown in Figure1.

The first region \(R_{0,1}\) is bounded by the \(y\) axis and the first curve \(f_1(y, t) = 0\) of the family (24).

5.2 Interior Curves

The Boundary Curves are a subset of a continuous family defined as:
\[
f(y, t) = I_1(y, t) - I_1(0, t_0) = I_1(0, t_0) - I_1 = c(t_0)
\]
which intersect the Critical Line at \(t=t_0\), where \(t_0\) varies from 0 to infinity.
f(y,t) = c(t_0) is a Boundary Curve when \( t_0 = \rho_i \), a root of \( \xi(s) \), otherwise it is an Interior Curve of the region \( R_{i,i+1} \) with \( \rho_i < t_0 < \rho_{i+1} \).

Let \( \alpha(0,t) = Rl[\xi(s)] \) on the Critical Line. From Equation(16)

\[
\alpha(0,t) = \frac{1}{2} + U(0,t)I_1(0,t) = \frac{1}{2} - (t^2 + \frac{1}{4})I_1(0,t) \tag{26}
\]

Since

\[
U(0,t) = -(t^2 + \frac{1}{4}) \quad \text{and} \quad I_{1*}(0,t) = 1/2(t^2 + \frac{1}{4})
\]

then

\[
\alpha(0,t) = (t^2 + \frac{1}{4})(I_{1*}(0,t) - I_1(0,t)) \tag{27}
\]

The chosen family of Interior Curves is,

\[
f(y,t) = I_{1*}(y,t) - I_1(y,t) = \alpha(0,t_0)/(t_0^2 + \frac{1}{4}) = c(t_0) \tag{28}
\]

The sign of \( (I_{1*}(y,t) - I_1(y,t)) \) is the same as the sign of \( \alpha(0,t_0) \). It is positive in the region \( R_{i,i+1} \) when \( i \) is even, and negative when \( i \) is odd.

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Figure 1. Subdivision of the Critical Strip by the Family of Boundary Curves.

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6 Zeros of $\xi(s)$ in the Critical Strip.

6.1 The interiors of the regions $R_{i,i+1}$ $i=0,1,2...$.

All points $(y,t)$ in the interior of a region $R_{i,i+1}$ lie on a curve of the family:

$$f(y,t) = I_{1*}(y,t) - I_1(y,t) = \alpha(0,t_0)/(t_0^2 + \frac{1}{4})$$  \hspace{1cm} (29)

The $t$ coordinate $t_0$ lies in the interval $\rho_i < t_0 < \rho_{i+1}$ and is not a zero of $\xi(s)$ on $y=0$. It follows that $I_{1*}(y,t) - I_1(y,t)$ cannot be zero on an Interior Curve, which, by Theorem 3.1, is a necessary condition for a zero of $\xi(s)$ to exist at $(y,t)$.

Hence there are no zeros of $\xi(s)$ in the interiors of all the regions $R_{i,i+1}$ $i=0,1,2,...$.

6.2 Points on the Boundary Curves.

With the notation $\alpha = Re[\xi(s)]$ and $\beta = Im[\xi(s)]$, Equations(16) are:

$$\alpha = \frac{1}{2} + UI_1 - VI_2 \quad \text{and} \quad \beta = VI_1 + UI_2.$$ \hspace{1cm} (30)

On a Boundary Curve where $f_i(y,t) = 0$,

$$I_{1*}(y,t) - I_1(y,t) = 0$$ \hspace{1cm} (31)

for $0 \leq y \leq \frac{1}{2}$ and $t = \rho_i$ when $y=0$.

With $I_{1*}$ in place of $I_1$ in Equations(30),

$$\alpha = \frac{1}{2} + UI_{1*} - VI_2 \quad \text{and} \quad \beta = VI_{1*} + UI_2.$$ \hspace{1cm} (32)

By elimination of $I_2$ and with the definition of $I_{1*}$ at (17), it follows that:

$$U(y,t)\alpha(y,t) + V(y,t)\beta(y,t) = 0.$$ \hspace{1cm} (33)

Then, with the definitions of $U$ and $V$ at (10),

$$\frac{\alpha(y,t)}{\beta(y,t)} = -\frac{V(y,t)}{U(y,t)} = \frac{2yt}{(t_0^2 - y^2 + \frac{1}{4})}.$$ \hspace{1cm} (34)

Equation(34) implies that $\alpha$ and $\beta$ are finite and of the same sign for $0 < y \leq \frac{1}{2}$ and $t > 0$.

At $y=1/2$ and $t=\tau_i$ on a Boundary Curve,

$$\beta(\frac{1}{2},\tau_i) = t.\alpha(\frac{1}{2},\tau_i)$$ \hspace{1cm} (35)
and the ratio $\beta/\alpha$ tends to infinity as $t$ tends to infinity.

Now
$$V_\alpha - U_\beta = (U^2 + V^2)(I_{2*}(y,t) - I_2(y,t))$$
(36)

then
$$\alpha(y,t) = 2yt(I_{2*}(y,t) - I_2(y,t))$$
(37)

and
$$\beta(y,t) = (t^2 - y^2 + \frac{1}{4})(I_{2*}(y,t) - I_2(y,t)).$$
(38)

$\alpha$ and $\beta$ have the same sign as $(I_{2*}(y,t) - I_2(y,t))$ on every Boundary Curve $f_i(y,t) = 0$. They are both positive when $i$ is even and negative when $i$ is odd. $(I_{2*}(y,t) - I_2(y,t))$ is finite, so that the conditions for a zero, given in Theorem 3.1, are not satisfied, except on the Critical Line.

6.3 Region $R_{0,1}$.

The lower boundary of region $R_{0,1}$ is not a member of the family $f_i(y,t) = 0$. It is the segment of the $y$ axis $0 \leq y \leq \frac{1}{2}$ with $t = 0$.

From (15),
$$\xi(y,0) = \frac{1}{2} + (y^2 - \frac{1}{4})I_1(y,0)$$
(39)

where, by calculation, $0.01152 \leq I_1(y,0) \leq 0.01155$, which proves that there are no zeros of $\xi(s)$ on the lower boundary of region $R_{0,1}$.

In Region $R_{0,1}$ the last interior curve to start from the $t$ axis intersects the Critical Line at $t=0$. Thereafter, Interior Curves start from points on the $y$ axis, $0 \leq y < \frac{1}{2}$, with $t=0$ and terminate on the boundary of the Critical Strip $y = \frac{1}{2}$.

The expression $I_{1*}(y,t) - I_1(y,t)$ is dominated by large values of $I_{1*}$ compared to small values of $I_1$, so that $I_{1*}(y,t) - I_1(y,t)$ is $> 0$ throughout Region $R_{0,1}$.

6.4 Conclusions from Section 6.

From a consideration of both Boundary Curves and Interior Curves, which cover the Critical Strip, it has been shown that there are no zeros of $\xi(s)$ in the Critical strip, except on the Critical Line.

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7 Conclusions.

A family of curves is defined which covers the entire Critical Strip. Particular members of this family, designated as Boundary Curves, intersect the Critical Line in the known roots of $\xi(s)$. All other curves of the family, referred to as Interior Curves, span the interiors of all the Regions $R_{i,i+1}, i = 0, 1, 2, \ldots$. By definition, all points on these curves do not satisfy the necessary and sufficient conditions for a zero of $\xi(s)$ to exist within the Critical Strip. Points on the Boundary Curves do satisfy a necessary condition for a zero to exist, but not sufficient conditions. It is shown that on a Boundary Curve, $Re[\xi(s)]$ and $Im[\xi(s)]$ are always finite and of the same sign, except at the known roots on the Critical Line, where they both become zero. The Riemann hypothesis is proved to be true.

References


